Fracture Statistics of torsion and dispersion relations in round bars

Gerardo Díaz

Departamento de Ciencia e Ingeniería de Materiales (IDIEIM), Facultad de Ciencias Físicas y Matemáticas, Universidad de Chile, Casilla 1420, Santiago, Chile.

Abstract
This paper has adopted a theoretical viewpoint for studying Fracture Statistics in round bars subjected to torsion, and for determining the cumulative probabilities of fracture using Weibull's specific-risk function for materials that exhibit volume and surface brittleness. The use of the defined-functions method has allowed to get the specific-risk-of-fracture function and, in addition, to carry out a separation between volume part and surface part concerning materials presenting both brittlenesses at the same time. Dispersion of the parameters are determined resorting to Fisher's information matrix.

1. INTRODUCTION
Statistical mechanics of fracture proposed by Weibull [1] is now being widely used for describing the fracture behaviour of brittle materials. Investigation has been directed to diverse aspects such as: the foundations of said mechanics [2, 3]; the application thereof for getting the cumulative probability of fracture concerning sundry materials subjected to different states of stress as for instance biaxial [4] and multiaxial [5] states, torsion [6-9], bending [10]; the evaluation of Weibull's parameters [11, 12]. This work is endeavouring to achieve the theoretical study of the torsion problem determining therefor the cumulative probabilities of fracture, the parameters of the specific-risk-of-fracture function, and the dispersion of the same.

2. STATISTICS OF FRACTURE THROUGH TORSION

2.1. Volume brittleness
The cumulative probability of fracture F(r) for materials with volume brittleness and subjected to some uniaxial state of shear stress is as follows according to Weibull's theory:

\[ F(r) = 1 - \exp \left\{ -\frac{1}{V_0} \int_V \phi [\tau(r)] \, dV \right\} \] (1)

where \( V_0 \) is the volume unit, \( V \) is the body volume, \( r \) is the position vector, \( \tau \) is the maximum shear-stress reached in the material before breaking, \( \tau(r) \leq \tau \) is the uniaxial stress-field, and \( \phi (\tau) \) is the specific-risk-of-fracture function. Weibull [1] has proposed the following analytical form for this function:

\[ \phi (\tau) = \begin{cases} (\frac{\tau - \tau_o}{\tau_i})^m & \tau \geq \tau_i \\ 0 & \tau < \tau_i \end{cases} \] (2)

where \( \tau_o \) and \( m \) are parameters depending on the manufacturing process of the material, while \( \tau_i \) is the stress under which there is no fracture.

In accordance with the elemental theory of torsion, the stress field, in cylindrical coordinates, for a round bar exhibiting volume brittleness and \( L \) in length and \( r \) in radius, may be expressed as follows:

\[ \tau(\rho) = \frac{M}{\pi r^3} \] (3)

where \( M \) is the torsional moment acting on the bar. The rewriting of equation (1) gives:

\[ \xi (\tau) = \ln \frac{1}{1 - F(\tau)} = \frac{1}{V_0} \int_V \phi [\tau(r)] \, dV \] (4)

If \( \phi (\tau) \) is given by equation (2) - i.e. using the defined functions method and considering the above equations (3) and (4) we get:

\[ \xi (\tau) = \frac{2\pi Lr^2}{V_0 (m+2)} \tau \left[ 1 + \frac{1}{m+1} \frac{\tau_i}{\tau} \right] \left( \frac{\tau - \tau_o}{\tau_i} \right)^{m+1} \] (5)
If \( n = 0 \) in equation (2), the consideration of equations (3) and (4) yields:

\[
\xi(r) = \frac{2\pi L r^2}{S_0 (m+2)} \left( \frac{r}{r_0} \right)^m
\]  

(6)

Weibulls' parameters \( m, r_0 \) and \( r \) may be obtained from equation (5) inasmuch as \( \xi(r) \) is known from the tests. If \( n = 0 \) a Wibull diagram may be plotted, and parameters \( m \) and \( r_0 \) may be obtained using the equation (6).

2.2. Surface brittleness

In the case of materials with surface brittleness, according to Weibull's theory the cumulative probability of fracture \( F(r) \) is given by an expression similar to equation (1) and that may be written in the following way, as equation (4):

\[
\xi(r) = \ln \frac{1}{1 - F(r)} = \frac{1}{S_0} \int_S \phi(\sigma) \, d\sigma
\]  

(7)

where \( S_0 \) is surface unit and \( S \) is the surface of the material.

In accordance with the elemental theory of torsion, the stress field for a round bar exhibiting surface brittleness and \( L \) in length and \( r \) in radius may be expressed as follows:

\[
\tau(\rho=r) = \tau
\]

\[
0 \leq \theta \leq 2\pi ; 0 \leq z \leq L
\]  

(8)

If \( \phi(\sigma) \) is given by Weibull's function, namely equation (2), then the consideration of equations (7) and (8) yields:

\[
\xi(r) = \frac{2\pi L r^2}{S_0} \left( \frac{r - r_i}{r_0} \right)^m
\]  

(9)

If \( \tau = 0 \) in equation (2), then equations (7) and (8) allow to get

\[
\xi(r) = \frac{2\pi L r^2}{S_0} \left( \frac{r}{r_0} \right)^m
\]  

(10)

Equations (9) and (10) permit the plotting of the respective Weibull diagrams for getting the parameters \( m, r_0 \) and \( r_i \), when \( r_i \neq 0 \) and \( r_i = 0 \), respectively.

2.3. Combined volume and surface brittleness

In the case of torsion applied to some round bar presenting volume and surface brittleness, the respective specific-risk-of-fracture functions may be obtained in a separate manner. Considering equations (5) and (9) we have:

\[
j_1 = \frac{2\pi L r^2}{V_0} \phi_v(\tau) + \frac{2\pi L r^2}{V_0} \phi_s(\tau)
\]

\[
\phi_v(\tau) = \frac{\tau_{0v}}{\tau} \left[ 1 + \frac{1}{m_v+1} \frac{\tau_{0v}}{\tau} \right] \left( \frac{\tau - \tau_{0v}}{\tau_{0v}} \right)^{m_v+1}
\]

(11)

\[
\phi_s(\tau) = \left( \frac{\tau - \tau_{0s}}{\tau_{0s}} \right)^{m_s}
\]

and if we take two groups of samples with \( L_i \) and \( r_i \), where \( i = 1, 2 \), then we get the following equations:

\[
\xi_1(r) = \frac{2\pi L_i r^2}{V_0} \phi_v(\tau) + \frac{2\pi L_i r^2}{V_0} \phi_s(\tau)
\]

\[
\xi_2(r) = \frac{2\pi L_2 r^2}{V_0} \phi_v(\tau) + \frac{2\pi L_2 r^2}{V_0} \phi_s(\tau)
\]  

(12)

This system has a non-trivial solution, because if \( r_i \neq r_j \), the associated determinant is not null, and the system is linearly independent. Hence the resolution of equation (12) for getting \( \phi_v \) and \( \phi_s \) yields:

\[
\phi_v(\tau) = \frac{V_0}{2\pi (r_1 - r_2)} \left[ \frac{\xi_1(\tau)}{L_1 r_1} - \frac{\xi_2(\tau)}{L_2 r_2} \right]
\]

(13)

\[
\phi_s(\tau) = \frac{S_0}{2\pi (r_1 - r_2)} \left[ \frac{r_1}{L_2 r_2} \xi_2(\tau) - \frac{r_2}{L_1 r_1} \xi_1(\tau) \right]
\]

In this way it has been possible to separate both functions of the specific risk of fracture, when the material is exhibiting volume brittleness and surface brittleness at the same time. Moreover the parameters of the respective specific-risk-of-fracture functions, for volume and surface brittleness, may be evaluated by preparing a non dimensional nomogram [12]. In the case of volume brittleness, considering \( \phi_v \) of the equations (11) and (13), we get

\[
\ln \phi_v(\tau) = \ln \left\{ \left[ 1 + \frac{\tau_{0v}}{\tau_{0v}} \right] \frac{\tau_{0v}}{m_v+1} \left( \frac{\tau}{\tau_{0v}} - 1 \right)^{m_v+1} \right\} + \ln \left( \frac{\tau_{0v}}{\tau_{0v}} \right)^{m_v}
\]

Then plotting

\[
\ln \left\{ \left[ 1 + \frac{\tau_{0v}}{m_v+1} \right] \frac{\tau_{0v}}{m_v+1} \left( \frac{\tau}{\tau_{0v}} - 1 \right)^{m_v+1} \right\}
\]

(15)
against in (τ/τ_r) for several values of Weibull's parameter m_v we obtain the nomogram, and the best fits the experimental points on the nomogram curve allows to obtain m_v, τ_v, and τ_r. In the case of surface brittleness, considering the equations (11) and (13), we get

\[ \ln \phi_s(\tau) = m_s \ln \left( \frac{\tau}{\tau_{0s}} - 1 \right) + \ln \left( \frac{\tau_s}{\tau_{0s}} \right)^{m_s} \]  \hspace{1cm} (16)

Now if we plot m_s ln (τ/τ_{0s}) against ln (τ/τ_{0s}) for various values of Weibull's modulus m_s we obtain a non-dimensional graph, and in the same way as explained above we obtained m_s, τ_{0s} and τ_{0s}.

3. DISPERSIONS OF THE PARAMETERS

The dispersion of the parameters of the cumulative-probability-of-fracture functions may be estimated through Fisher's information matrix [13]. The coefficients of the Fisher matrix are determined using the following relation-ship:

\[ r_{ij} = -n \int \left[ \frac{n^2 \ln(\tau; \theta)}{0 \theta; 0 \theta} \right] f(\tau) d\tau \]  \hspace{1cm} (17)

where r_{ij} is the coefficient i, j, n is sample size, \theta are the parameters, and \( f(\tau) = \frac{dF(\tau)}{d\tau} \) is the density function of fracture probability. If the specific-risk-of-fracture function is a Weibull function with \( \tau = 0 \) then the Fisher matrix elements are as follows:

\[ r_{11} = n \left( \frac{1}{g} \frac{ag}{am} \right)^2 + \frac{2n}{gm} \frac{ag}{om} (0.42277 - \ln g) \]
\[ = \frac{n}{m^2} \left( 1.82379 - 0.84555 \ln g + \ln^2 g \right) \]
\[ r_{12} = \frac{n}{\tau_o} \left( \ln g - \frac{m}{g} \frac{ag}{am} - 0.42277 \right) \]
\[ r_{22} = \frac{n}{\tau_o} \left( \frac{m}{g} \frac{ag}{am} \right)^2 \]  \hspace{1cm} (18)

where \( g = g(m) \) for both cases of brittleness is given by:

\[ g_v(m) = \frac{2\pi Lr^2}{V_0(m+2)} ; \frac{ag}{am} = \frac{2\pi Lr^2}{V_0(m+2)} \]
\[ g_s(m) = \frac{2\pi Lr}{S_0} ; \frac{ag}{am} = 0 \]  \hspace{1cm} (19)

Hence if we consider only the cases where \( \tau = 0 \) as well as the respective equations (6) and (10) for volume brittleness and surface brittleness, then the matrix of variances and covariances is easily obtained through the inversion of the Fisher matrix, with the condition r_{11} \geq 0. Therefore the variances and covariances are determined as follows:

\[ \text{Var}(m) = \frac{r_{22}}{r_{11}} \]
\[ \text{Var}(\tau_o) = \frac{r_{11}}{r_{11} \tau_o} \frac{r_{12}}{r_{12} \tau_o} \]
\[ \text{Co-Var}(m, \tau_o) = \frac{r_{12}}{r_{11} \tau_o} \frac{r_{12}}{r_{12} \tau_o} \]  \hspace{1cm} (20)

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