Obtention of the specific-risk-of-fracture function of a square prismatic bar subjected to torsion

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Summary

The cumulative probabilities of fracture and the specific-risk-of-fracture functions are obtained for volume and surface brittlenesses of a square prismatic bar. In order to solve the problem analytically the square prismatic bar is assumed to be included within a cylindrical matrix made of the same material as the bar. When the specific-risk-of-fracture function is unknown, it is obtained using the method of integral equations, and when the material of the bar is simultaneously exhibiting volume and surface brittlenesses, both can be separated. Besides, the impossibility of reducing the integral equations to one having an exact analytical solution is shown, inasmuch as the kernel of such equations is infinite. The dispersion of the parameters of the specific-risk-of-fracture functions is determined using the Fisher information matrix.

1. INTRODUCTION

The problem of the torsion of prismatic bars having a cross-section that is not circular is complicated even when pure states of torsion are being considered, i.e. when the cross-sections of the bars are exhibiting only a torsional moment and, for instance, there is no flexure in the bars. Díaz and Morales have determined in a recent work [1] the functions of cumulative probability of fracture in cylindrical glass bars subjected to a state of pure torsion [2-6]. In that case the problem was less complicated because the hypothesis of the invariability of plane cross-sections was allowing to simplify the same. However, strictly speaking such hypothesis is not valid in the torsion of prismatic bars whose cross-section is not circular; the cross-sections of the bar become warped and hence the stress-field in such a section is different. In view of the complexity of torsion problem, the same is to be dealt with using the theory of elasticity, which solves this problem by resorting to indirect methods such as the method of membrane analogy, for instance [7]. In the method of the membrane, independently of the cross-section of the bar, the torsion problem thereof leads us to the same differential equation corresponding to the problem of the equilibrium of some membrane stretched over a contour that is exhibiting the same geometry as the cross-section of the bar and that is being subjected to a uniformly distributed pressure. Up to now the problem of determining torsion stresses of some bar whose cross-section is arbitrary has been solved, in a first approximation, replacing cross-section contour by the inscribed circle contour, thus leaving aside those cross-sectional zones located in the vicinity of the vertices and whose stresses are very reduced and of little influence on bar strength.

The purpose of this work is, first, obtaining the specific-risk-of-torsional-fracture function of a square prismatic bar considering therefor volume and surface brittlenesses, and, secondly, determining the dispersion of bar parameters. In order to be able to solve the problem in an analytical way we are proposing to include the square prismatic bar within a cylinder, i.e. to circumscribe a circle in the square section. As concerns experimentation it is necessary that the square bar and the circumscribed cylinder must be of the same material and, as far as possible, be bonded using some epoxy resin. Of course the stress field thus obtained is not the same as that of the bar considered without the cylindrical matrix; however, this is a method allowing to get the volume and surface functions of the specific risk of fracture, without resorting to the numerical methods and using therefor the stress field that is acting within some prismatic bar of circular cross-section, solving the problem in a purely analytical way. In this proposed approximation we shall disregard the interphase generated between the square bar and the circumscribed cylinder. Two extreme situations are to be distinguished in the experiences carried out, namely: glassy materials and opaque materials subjected to the torsion test. For the glassy materials the cylindrical matrix is to be made of the same material as the square bar and through sintering sol as to get a transparent matrix and then control the experiences using some optical method and consider only the fractures that are starting in the volume and/or on the surface of the square prismatic bar. The other extreme situation
namely that of opaque materials, obliges to control the experiences using therefor some acoustic emission apparatus to the end of detecting the factures that are starting in the cylindrical matrix and in the square bar; and this detection will be possible owing to the reflection of the acoustic wave on the interphase between the square bar and the circumscribed matrix.

2. VOLUME BRITTLENESS

According to the probabilistic strength of materials, the cumulative probability of fracture of some homogeneous and isotropic material exhibiting volume brittleness and subjected to a varying uniaxial stress fields, is given by the following expression [8-12]:

\[
F(r) = 1 - \exp \left\{ - \frac{1}{V} \int \phi \left[ \tau(r) \right] \, dV \right\}
\]  

(1)

where \( V \) is the volume of the material, \( V \) is the unit volume, \( r \) is the position vector, \( \tau(r) \leq \tau \) is the stress field, and \( \phi(\tau) \) is the specific-risk-of-fracture function, also known as Weibull’s function. In general this function is exhibiting the following analytical expression:

\[
\phi(\tau) = \begin{cases} 
(\frac{\tau - \tau_i}{\tau_0})^m & \tau \geq \tau_i \\
0 & \tau < \tau_i 
\end{cases}
\]

(2)

where \( \tau_i \) is the stress below which there is no fracture, i.e., the stress for which the probability of fracture is null, \( m \) and \( \tau_0 \) are parameters that depend on the manufacturing process of the material.

The stress field of some bar of round section, having one end restrained and whose free end is subjected to torsion moment \( M \) is given by

\[
\tau(\rho) = \frac{\rho}{r} \tau \leq \tau = \frac{2M}{\pi r^2}
\]

(3)

where \( \rho \) is the radius and \( L \) is the length of the bar. Let us consider a square prismatic bar, of side \( a \), inscribed within a cylindrical matrix of the same material as this bar; then, in accordance with the proposed model the radius of this circumscribed cylinder is \( r = a/\sqrt{16} \). Hence the expression of the stress field within the square kern of the cylindrical bar is:

\[
\tau(\rho) = \frac{\sqrt{2}}{a} \rho \tau \quad 0 \leq 0 \leq \pi/4 ; \quad 0 \leq \rho \leq r
\]

(4)

Equation (1) can be rewritten as follows [13]:

\[
\xi(\tau) = \ln \frac{1}{1 - F(\tau)} = \frac{1}{V} \int \phi \left[ \tau(r) \right] \, dV
\]

(5)

Using the method of the defined functions, i.e., the method that is postulating a potential form of the specific-risk-of-fracture function, we can determine the cumulative probability of volume fracture through torsion. If the specific-risk-of-fracture function \( \phi(\tau) \) is expressed by means of equation (2), the consideration of equations (4) and (5) yields:

\[
\xi(\tau) = \frac{4L\tau^2}{V_0(m+1)(m+2)} \left( \frac{\tau_0}{\tau} \right)^m \left( \frac{\tau_0}{\tau} \right)^2 \int_0^{\pi/4} \int_0^{\tau/\sqrt{2}} \cos \theta \, d\theta \int_0^{\pi/4} \phi(\eta) \eta \, d\eta
\]

(6)

If \( \tau_i = 0 \) we get:

\[
\xi(\tau) = \frac{L\tau^2}{V_0} \left( \frac{\tau_0}{\tau} \right)^m \int_0^{2^{-m/2}} \int_0^{\pi/4} \phi(\eta) \eta \, d\eta
\]

(7)

Weibull’s parameters \( m \), \( \tau_0 \), and \( \tau_i \) can be obtained from above equation (6) inasmuch as \( \xi(\tau) \) is known through the tests. If \( \tau_i = 0 \) a Weibull diagram can be plotted, and then the parameters \( m \) and \( \tau_0 \) and \( \tau_i \) can be obtained using equation (7).

If a known analytical form is not postulated for the specific-risk-of-fracture function, then equations (4) and (5) allow to get the following integral equation wherein \( \phi(\tau) \) is the unknown function [14-16]:

\[
\xi(\tau) = \frac{4L\tau^2}{V_0} \int_0^{\pi/4} \int_0^{\tau/\sqrt{2}} \cos \theta \, d\theta \int_0^{\pi/4} \phi(\eta) \eta \, d\eta
\]

(8)

This integral equation (8) can be rewritten as follows:

\[
\xi(\tau) = \frac{2\sqrt{2} L\tau^2}{V_0} \int_0^{\tau} \frac{\Phi(\xi) \, d\xi}{\sqrt{\xi^2 - (\tau/\sqrt{2})^2}}
\]

(9a)

\[
\phi(\xi) = \frac{1}{\xi} \int_0^{\xi} \phi(\eta) \eta \, d\eta
\]

(9b)
Equation (9a) is a new integral equation wherein the unknown function is \( \phi \), which once solved and introduced into equation (9b), allows solving the problem, i.e., getting the function \( \phi(\tau) \). But equation (9a) has no known analytical solution, its kernel is infinite and hence equation (8) too has no known analytical solution. By directly operating on equation (8) it is possible to obtain its solution by means of Taylor series expansions. Therefore, the solution for \( \phi(\tau) \) is:

\[
\phi(\tau) = \frac{V_0}{La^2} \sum_{n=0}^{\infty} \left( \frac{n+2}{2^n \cdot \xi^{(n)(0)}} \right) \tau^n
\]

(10)

3. SURFACE BRITTLENESS

In the case of materials exhibiting surface brittleness, the cumulative probability of fracture is given by the following expression:

\[
F(\tau) = 1 - \exp \left\{ -\frac{1}{S} \int \phi(\tau) dS \right\}
\]

(11)

where \( S \) is the surface of the material, and \( S_o \) is the unit surface. Similarly to equation (5), above equation (11) can be rewritten as follows:

\[
\xi(\tau) = \frac{\ln \left( \frac{1}{1 - F(\tau)} \right)}{S} \int \phi(\tau) dS
\]

(12)

In accordance with the model that has been proposed, namely a square prismatic bar inscribed within a cylindrical bar, the stress field, considering all the sides of the square bar, is given by:

\[
\tau(\rho) = \begin{cases} 
\sqrt{1 + \left( \frac{2x}{a} \right)^2} \frac{\tau}{\sqrt{2}} & 0 \leq Z \leq L; 0 \leq X \leq a/2; y = a/2 \\
\sqrt{1 + \left( \frac{2y}{a} \right)^2} \frac{\tau}{\sqrt{2}} & 0 \leq Z \leq L; 0 \leq y \leq a/2; x = a/2 \\
\frac{\sqrt{2}}{a} \rho & 0 \leq \rho \leq \pi/4; 0 \leq \rho \leq a/2 \cos 0; Z = L
\end{cases}
\]

(13)

The torsion moment \( M \) is acting in the \((X,Y)\) plane. Hence, if \( \phi(\tau) \) is expressed by means of equation (2), the consideration of equations (12) and (13) yields:

\[
\xi(\tau) = \frac{4\sqrt{2} aL}{S_o} \left( \frac{\tau}{\tau_o} \right)^m \frac{\tau_x}{\sqrt{2} \tau_o} \int \frac{(\eta-1)^m \eta d\eta}{\eta^2 - (\tau/\sqrt{2})^2} + \frac{4a^2}{S_o(m+1)(m+2)} \left( \frac{\tau}{\tau_o} \right)^m \left( \frac{\tau_x}{\sqrt{2} \cos 0} \right)^2
\]

(14)

\[
\int_0^{\pi/4} \left[ 1 + \left( \frac{\tau}{\tau_o} \right)^m \right] \left( \frac{\tau_x}{\sqrt{2} \cos 0} - 1 \right)^{m+1} d\tau
\]

(15)

Weibull’s parameters \( m, \tau_o \), and \( \tau \) can be obtained from above equation (14) inasmuch as \( \xi(\tau) \) is known through the tests. If \( \tau = 0 \) a Weibull diagram can be obtained using equation (15).

If a known analytical form is not postulated for the specific-risk-of-fracture function, in the same fashion as for the case of volume brittleness, there is obtained the following equation wherein \( \phi(\tau) \) is the unknown function for the present instance of surface brittleness:

\[
\xi(\tau) = \frac{4\sqrt{2} aL}{S_o} \left( \frac{\tau}{\tau_o} \right)^m \frac{\tau_x}{\sqrt{2} \tau_o} \int \frac{(\eta-1)^m \eta d\eta}{\eta^2 - (\tau/\sqrt{2})^2} + \frac{4a^2}{S_o(m+1)(m+2)} \left( \frac{\tau}{\tau_o} \right)^m \left( \frac{\tau_x}{\sqrt{2} \cos 0} \right)^2
\]

(16)

This integral equation (16) can be rewritten as follows:

\[
\xi(\tau) = \frac{4\sqrt{2} aL}{S_o} \left( \frac{\tau}{\tau_o} \right)^m \frac{\tau_x}{\sqrt{2} \tau_o} \int \frac{(\eta-1)^m \eta d\eta}{\eta^2 - (\tau/\sqrt{2})^2} + \frac{4a^2}{S_o(m+1)(m+2)} \left( \frac{\tau}{\tau_o} \right)^m \left( \frac{\tau_x}{\sqrt{2} \cos 0} \right)^2
\]

(17a)

\[
\phi(\eta) = \eta \phi(\eta) + \frac{a}{2L} \int_0^\eta \phi(\xi) d\xi
\]

(17b)

Above equation (17a) is of the same type as equation (9a), so that here applies the same comments as for that equation. Therefore, by directly operating on equation (16) and by means of some Taylor series expansions, the solution for \( \phi(\tau) \) is:

\[
\phi(\tau) = \frac{S_o}{La} \sum_{n=0}^{\infty} \left[ \frac{1}{2(n+2)} \int_0^{\pi/4} \frac{d\eta}{\cos n+20} \right] \left( \frac{\tau}{\tau_o} \right)^m \left( \frac{\tau_x}{\sqrt{2} \cos 0} \right)^2
\]

(18)

4. JOINT VOLUME AND SURFACE BRITTLENESSES

For square prismatic bars subjected to torsion and exhibiting joint volume and surface brittlenesses, the respective specific-risk-of-fracture func-
tions can be separately obtained. This can be achieved using the method of integral equations. The consideration of equations (9a), (9b), (17a) and (17b) yields:

\[\xi(\tau) = \xi_v(\tau) + \xi_s(\tau)\]  

\[
\xi_v(\tau) = \frac{2\sqrt{2} \sqrt{La^2}}{V_0} \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_v(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} \]  

\[
\xi_s(\tau) = \frac{4\sqrt{2} aL}{S_0} \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_s(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} \]  

If we take two groups of samples similarly made using the same material, and having the dimensions \(L_i\) and \(a_i\), where \(i = 1,2\), then we get a system of linear equations having a non-trivial solution, because of \(a_i \neq a_i\), the associated determinant is not null and the system if linearly independent. There is obtained the following pair of integral equations wherein the unknown functions are \(\phi_v\) and \(\phi_s\):

\[
\frac{V_0}{2\sqrt{2} (a_1 - a_2)} \left[ \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_v(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} \right] - \frac{1}{V_0} \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_s(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} = 0 

\]

\[
\frac{S_0}{4\sqrt{2} (a_1 - a_2)} \left[ \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_s(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} \right] - \frac{1}{S_0} \int_{\tau/\sqrt{2}}^{\tau} \frac{\phi_v(\xi) d\xi}{\sqrt{\xi^2 - (\tau / \sqrt{2})^2}} = 0 

\]

These two equations (20a) and (20b) constitute the solution to the problem of the separation of the volume and surface functions of the specific risk of fracture. Solving integral equation (9b) we obtain the specific-risk-of-volume-fracture function, namely

\[\phi_v(\tau) = \frac{1}{\tau} \frac{d}{d\tau} \left[ \tau \phi_v(\tau) \right] \]  

where \(\phi_v(\tau)\) is obtained from equation (20a). In order to get the specific-risk-of-surface-fracture function it is necessary to solve the integral equation given by (17b) and its solution is:

\[\phi_s(\eta) = \frac{1}{\tau + \frac{a}{\tau}} \int_{0}^{\tau} \frac{d}{d\eta} \left[ \eta \phi_s(\eta) \right] \eta^\frac{m}{2} d\eta \]  

where \(\phi_s(\eta)\) is known from equation (20b).

Hence, for equations (20a) and (20b) the only alternative left to us is resorting to the numerical methods in order to thus separate the two specific-risk-of-fracture functions.

5. PARAMETERS DISPERSION

Fisher's information matrix [17] is determined using the following expression:

\[\tau_{ij} = -n \mathbf{E} \left( \frac{df(\tau; m, \tau_o)}{d\tau} \right) \]  

\[\{\theta\} = \{m, \tau_o\} \]  

where the operator \(\mathbf{E}\) is the expected value, \(n\) is the number of samples tested, and \(f(\tau)\) is the density function given by

\[f(\tau) = \frac{df(\tau)}{d\tau} = k \frac{m}{\tau_o} \left( \frac{\tau}{\tau_o} \right)^{m-1} \exp \left\{ -k \left( \frac{\tau}{\tau_o} \right)^m \right\} \]  

where \(k\) has the following expressions for the respective volume and surface brittlenesses:

\[k_v = \frac{La^2}{V_0} \frac{2^{1-m/2}}{m+2} \int_0^{\pi/4} \frac{d\theta}{\cos^{m+2}\theta} \]  

\[k_s = \frac{La^2}{S_0} \frac{2^{2-m/2}}{1 + \frac{a/L}{2(m+2)}} \int_0^{\pi/4} \frac{d\theta}{\cos^{m+2}\theta} \]

The elements of the Fisher matrix are:

\[r_{11} = n \left( \frac{1}{k \omega m} \right)^2 + 2n \frac{ak}{km} \left( 0.42277 - \ln k \right) \]  

\[r_{12} = n \frac{a}{k} \left( 0.42277 - \ln k + 0.42277 \right) \]  

\[r_{22} = n \left( \frac{m}{k} \right)^2 \]

Therefore, is we consider only the cases \(\tau_i = 0\), of equations (7) and (15), for volume and surface brittlenesses respectively, then the matrix of variances and covariances is easily obtained through the inversion of the Fisher matrix, with the condition that \(r_{qq} \geq 0\). Finally the variances and covariances are determined using the following expressions:

\[\text{Var}(m) = \frac{r_{11}}{r_{11} r_{22} - r_{12}^2} \]

\[\text{Var}(\tau_o) = \frac{r_{11}}{r_{11} r_{22} - r_{12}^2} \]

\[\text{Covar}(m, \tau_o) = \frac{r_{12}}{r_{11}^2 - r_{12}^2} \]  

\[\text{(27)} \]
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